Heteroclinic orbit and tracking attractor in cosmological model with a double exponential potential

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In this paper, the dynamical heteroclinic orbit and attractor have been employed to make the late-time behaviors of the model insensitive to the initial condition and thus alleviates the fine tuning problem in cosmological dynamical system of barotropic fluid and quintessence with a double exponential potential. The late-time asymptotic behavior of the double exponential model does not always correspond to the single case. The heteroclinic orbits are the non-stationary solutions and in general they will interpolate between the critical points. Although they can not be shown analytically, yet a numerical calculation can reveal most of their properties. Varied heteroclinic orbits and attractors including tracking attractor and de Sitter attractor have been found.

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I. INTRODUCTION

Astronomical observation on the cosmic microwave background(CMB) anisotropy[1], supernova type Ia(SNIa)[2] and SLOAN Digital Sky Survey(SDSS)[3] depicted that our Universe is spatially flat, with about seventy percent of the total density resulting from dark energy that has an equation of state $w < -1/3$ and accelerates the expansion of the Universe. Several candidates to represent dark energy have been suggested and confronted with observation: cosmological constant, quintessence with a single field[4] or with N coupled field[5], phantom field with canonical[6] or Born-Infield type Lagrangian[7], k-essence[8] and generalized Chaplygin gas(GCG)[9].

One of the most important issues for dark energy models is the fine tuning problem, and a good model should limit the fine tuning as much as possible. The dynamical attractor of the cosmological system has been employed to make the late time behaviors of the model insensitive to the initial condition of the field and thus alleviates the fine tuning problem. In quintessence[10] and phantom[11] models, the dynamical systems have tracking attractors that make the quintessence and phantom evolve by tracking the equation of state of the background cosmological fluid so as to alleviating the fine tuning problem. In addition, there are also two late time attractors in the phantom system corresponding to the big rip phase[12] and de Sitter phase[13]. On the other hand, exponential potentials can arise from string/M-theory, e.g.via compactification on product spaces possibly with fluxes. In this case, the equations of motion can be written as an autonomous system, and the power-law and de Sitter solutions can be determined by an algebraic method. The properties of the attractor solutions of exponential potentials can lead to models of quintessence [14]. And general exact solution for double exponential potential with one exponent is the negative of the other for quintessence was presented in Ref.[15]. The aim of this paper is to study the cosmological dynamics of barotropic fluid and scalar field with a double exponential potential and point out that the late-time asymptotic behavior does not always corresponding to the single-exponential case[16]. We show that the existence of tracking attractor and de Sitter attractor. We also find some heteroclinic orbits which mean solutions interpolate between different critical points. Emphasis must be placed on that the tracking orbits have the similar but not exactly equal dynamical behavior and the initial possibilities consist in a wide range.

II. PHASE SPACE AND CRITICAL POINTS

We consider 4-dimensional gravity with barotropic fluid and a scalar ϕ which only depend on cosmic time t. The scalar has a double exponential potential

$$V(\phi) = \lambda_1 e^{-\alpha_1 \phi} + \lambda_2 e^{-\alpha_2 \phi}. \tag{1}$$

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Since current observations favor flat Universe, we will work in the spatially flat Robertson-Walker metric. The corresponding equations of motion and Einstein equations could be written as

$$\begin{aligned}
\dot{H} &= -\frac{\kappa^2}{2}(\rho_\gamma + p_\gamma + \dot{\phi}^2), \\
\dot{\rho_\gamma} &= -3H(\rho_\gamma + p_\gamma), \\
\ddot{\phi} &+ 3H\dot{\phi} + V'(\phi) = 0, \\
H^2 &= \frac{\kappa^2}{3}(\rho_\gamma + \rho_\phi),
\end{aligned} \tag{2}$$

where $\kappa^2 = 8\pi G$, ρ_γ is the density of fluid with a barotropic equation of state $p_\gamma = (\gamma - 1)\rho_\gamma$, where $0 \leq \gamma \leq 2$ is a constant that relates to the equation of state by $w = \gamma - 1$. The overdot represents a derivative with respect to t, the prime denotes a derivative with respect to ϕ. $\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and $p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$ are the energy density and pressure of the ϕ field respectively, and H is the Hubble parameter. Phase space methods are particularly useful when the equations of motion are hard to solve analytically for the presence of barotropic density. In fact, the numerical solutions with random initial conditions are not a satisfying alternative because of these may not reveal all the important properties. Therefore, combining the information from the critical points analysis with numerical solutions, one is able to give the complete classification of solutions according to their late-time behavior. Similarly as in Ref.[10, 16], we introduce the following dimensionless variables $x = \frac{\kappa}{\sqrt{6}H}\dot{\phi}$, $y = \frac{\kappa\sqrt{\lambda_1 e^{-\alpha_1\phi}}}{\sqrt{3}H}$, $z = \frac{\kappa\sqrt{\lambda_2 e^{-\alpha_2\phi}}}{\sqrt{3}H}$, $\Gamma = \frac{V(\phi)V''(\phi)}{V'^2(\phi)}$ and $N = \log a$. Then, the Eqs.(2) could be reexpressed as the following system of equations:

$$\begin{aligned}
\frac{dx}{dN} &= \frac{3}{2}x[\gamma(1 - x^2 - y^2 - z^2) + 2x^2] - [3x - \frac{1}{\kappa}\sqrt{\frac{3}{2}}(\alpha_1 y^2 + \alpha_2 z^2)], \\
\frac{dy}{dN} &= \frac{3}{2}y[\gamma(1 - x^2 - y^2 - z^2) + 2x^2] - \frac{1}{\kappa}\sqrt{\frac{3}{2}}\alpha_1 xy, \\
\frac{dz}{dN} &= \frac{3}{2}z[\gamma(1 - x^2 - y^2 - z^2) + 2x^2] - \frac{1}{\kappa}\sqrt{\frac{3}{2}}\alpha_2 xz.
\end{aligned} \tag{3}$$

Also, we have a constraint equation

$$\Omega_\phi + \frac{\kappa^2\rho_\gamma}{3H^2} = 1, \tag{4}$$

where

$$\Omega_\phi = \frac{\kappa^2\rho_\phi}{3H^2} = x^2 + y^2 + z^2. \tag{5}$$

Different from the case in the single exponential potential, the parameter Γ is dependent on ϕ:

$$\Gamma = \frac{(\alpha_1^2 y^2 + \alpha_2^2 z^2)(y^2 + z^2)}{(\alpha_1 y^2 + \alpha_2 z^2)^2} \tag{6}$$

The equation of state for the scalar field could be expressed in terms of the new variables as

$$w_\phi = \frac{p_\phi}{\rho_\phi} = \frac{x^2 - y^2 - z^2}{x^2 + y^2 + z^2}. \tag{7}$$

According to the Eqs.(3), one can obtain the critical points and study the stability of these points. Substituting linear perturbation $x = x + \delta x$, $y = y + \delta y$, $z = z + \delta z$ near the critical points into the three independent equations, to first order in the perturbations, gives the evolution equations of the linear perturbations, from which we could yield three eigenvalues. Stability requires the real part of all eigenvalues to be negative. The results are contained in Table I.

Case	Critical points (x,y,z)	Ω_ϕ	w_ϕ	Stability
(i)	$\sqrt{\frac{3}{2}}\frac{\gamma\kappa}{\alpha_1}$,$\sqrt{\frac{3}{2}}\sqrt{\frac{(2\gamma-\gamma^2)\kappa^2}{\alpha_1^2}}$,0	$\frac{3\gamma\kappa^2}{\alpha_1^2}$	$\gamma-1$	stable
(ii)	$\sqrt{\frac{3}{2}}\frac{\gamma\kappa}{\alpha_2}$,0,$\sqrt{\frac{3}{2}}\sqrt{\frac{(2\gamma-\gamma^2)\kappa^2}{\alpha_2^2}}$	$\frac{3\gamma\kappa^2}{\alpha_2^2}$	$\gamma-1$	stable
(iii)	$\frac{\alpha_1}{\sqrt{6}\kappa}$,$\sqrt{1-\frac{\alpha_1^2}{6\kappa^2}}$,0	1	$-1+\frac{\alpha_1^2}{3\kappa^2}$	stable
(iv)	$\frac{\alpha_2}{\sqrt{6}\kappa}$,0,$\sqrt{1-\frac{\alpha_2^2}{6\kappa^2}}$	1	$-1+\frac{\alpha_2^2}{3\kappa^2}$	stable
(v)	0,$(1-\frac{\alpha_1}{\alpha_2})^{-1/2}$,$(1-\frac{\alpha_2}{\alpha_1})^{-1/2}$	1	-1	stable
(vi)	$\pm$1,0,0	1	1	unstable

TABLE I: The properties of the critical points.

Cases (i) and (ii): These are tracking attractors. The linearization of system (3) about these critical points yields three eigenvalues:

$$\Big[\frac{3(\alpha_1-\alpha_2)\gamma}{2\alpha_1},\frac{3\{\alpha_1^2(\gamma-2)-\sqrt{\alpha_1^2(\gamma-2)[\alpha_1^2(9\gamma-2)-24\gamma^2\kappa^2]}\}}{4\alpha_1^2},\frac{3\{\alpha_1^2(\gamma-2)+\sqrt{\alpha_1^2(\gamma-2)[\alpha_1^2(9\gamma-2)-24\gamma^2\kappa^2]}\}}{4\alpha_1^2}\Big]; \tag{8}$$

$$\Big[\frac{3(\alpha_2-\alpha_1)\gamma}{2\alpha_2},\frac{3\{\alpha_2^2(\gamma-2)-\sqrt{\alpha_2^2(\gamma-2)[\alpha_2^2(9\gamma-2)-24\gamma^2\kappa^2]}\}}{4\alpha_2^2},\frac{3\{\alpha_2^2(\gamma-2)+\sqrt{\alpha_2^2(\gamma-2)[\alpha_2^2(9\gamma-2)-24\gamma^2\kappa^2]}\}}{4\alpha_2^2}\Big]. \tag{9}$$

According to the eigenvalue (8), the stability of the attractor requires the condition $\sqrt{3\gamma}\kappa<\alpha_1<\alpha_2$ or $\alpha_2<\alpha_1<-\sqrt{3\gamma}\kappa$. For the eigenvalue (9), it requires the condition $\sqrt{3\gamma}\kappa<\alpha_2<\alpha_1$ or $\alpha_1<\alpha_2<-\sqrt{3\gamma}\kappa$.

Cases (iii) and (iv): These are quintessence attractors. The linearization of system (3) about these critical points yields three eigenvalues:

$$\Big[\frac{\alpha_1(\alpha_1-\alpha_2)}{2\kappa^2},\frac{\alpha_1^2}{2\kappa^2}-3,\frac{\alpha_1^2}{\kappa^2}-3\gamma\Big]; \tag{10}$$

$$\Big[\frac{\alpha_2(\alpha_2-\alpha_1)}{2\kappa^2},\frac{\alpha_2^2}{2\kappa^2}-3,\frac{\alpha_2^2}{\kappa^2}-3\gamma\Big]. \tag{11}$$

According to the eigenvalue (10), the stability of the attractor requires the condition $0<\alpha_1<\sqrt{3\gamma}\kappa$ and $\alpha_1<\alpha_2$ or $-\sqrt{3\gamma}\kappa<\alpha_1<0$ and $\alpha_1>\alpha_2$. For the eigenvalue (11), it requires the condition $0<\alpha_2<\sqrt{3\gamma}\kappa$ and $\alpha_2<\alpha_1$ or $-\sqrt{3\gamma}\kappa<\alpha_2<0$ and $\alpha_2>\alpha_1$.

Cases (v): The critical point is a attractor corresponding to equation of state $w=-1$ and cosmic energy density parameter $\Omega_\phi=1$, which is a de Sitter attractor. The condition of such a de Sitter attractor is $\alpha_1\alpha_2<0$. The linearization of system (3) about these critical points yields three eigenvalues:

$$\Big[-3\gamma,\frac{-3\kappa^2-\sqrt{12\alpha_1\alpha_2\kappa^2+9\kappa^4}}{2\kappa^2},\frac{-3\kappa^2+\sqrt{12\alpha_1\alpha_2\kappa^2+9\kappa^4}}{2\kappa^2}\Big]. \tag{12}$$

Cases (vi): These critical points corresponding to kinetic-dominated solutions in the asymptotic regime. The linearization of system (3) about these critical points yields three eigenvalues:

$$\Big[6-3\gamma,3-\sqrt{\frac{3}{2}}\frac{\alpha_1}{\kappa},3-\sqrt{\frac{3}{2}}\frac{\alpha_2}{\kappa}\Big]; \tag{13}$$

$$\Big[6-3\gamma,3+\sqrt{\frac{3}{2}}\frac{\alpha_1}{\kappa},3+\sqrt{\frac{3}{2}}\frac{\alpha_2}{\kappa}\Big]. \tag{14}$$

Although the kinetic-dominated critical points are always unstable, we will find some new properties in numerical analysis.

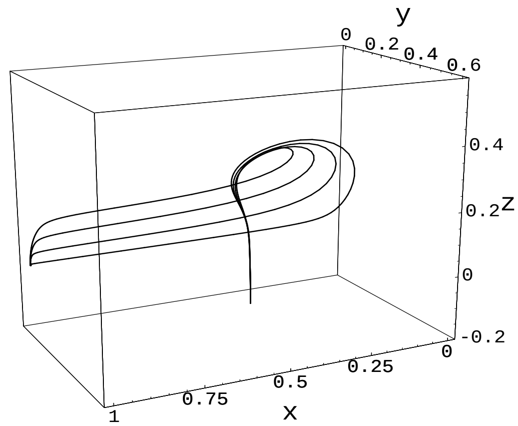

FIG. 1: The attractor property of the quintessence at the presence of dust matter. We choose $\alpha_1 = 2$, $\alpha_2 = \sqrt{5}$, $\gamma = 1$, $\kappa = 1$. The heteroclinic orbit connects the critical point which corresponds to the case (i) to the kinetic-dominated critical point $(1, 0, 0)$.

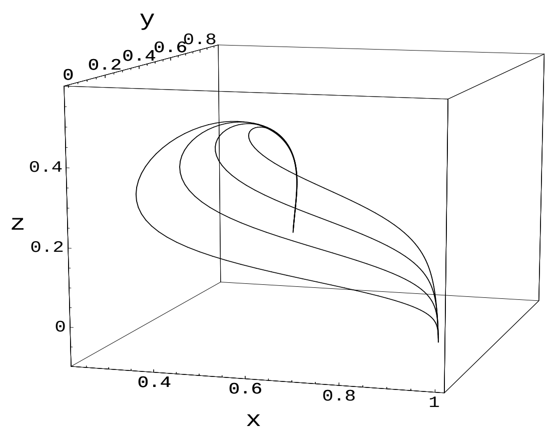

FIG. 2: The attractor property of the quintessence at the presence of dust matter. We choose $\alpha_1 = 1$, $\alpha_2 = 2$, $\gamma = 1$, $\kappa = 1$. The heteroclinic orbit connects the critical point which corresponds to the case (iii) to the kinetic-dominated critical point $(1, 0, 0)$.

III. HETEROCLINIC ORBITS

Critical points are always exact constant solution in the context of autonomous dynamical systems. These points are often the extreme points of the orbits and therefore describe the asymptotic behavior. If the solutions interpolate between critical points they can be divided into heteroclinic orbit and homoclinic orbit(closed loop). The heteroclinic orbit connects two different critical points and the homoclinic orbit is an orbit connecting a critical point to itself. If the numerical calculation is associated with the critical point analysis, then we will find all kinds of heteroclinic orbit, as show in Figure.1 to Figure.3. The initial point is $(\pm 1, 0, 0)$ for all orbits since it is a repeller. Especially, the heteroclinic orbit is shown in Figure 2, which connects the tracking attractor to $(1, 0, 0)$. In Figure 3, the heteroclinic orbit connects the de Sitter attractor to $(1, 0, 0)$. Tracking behavior consists in the possibility to avoid the fine tuning of initial conditions, obtaining the similar behavior for a while range of initial possibilities. If we take cosmic time $t = t_0$, the phase space contain a two-dimensional submanifold corresponding to set of initial possibilities. This initial submanifold consists with the intersection set of heteroclinic orbit and $t = t_0$ surface.

In Figure 4, we plot the dynamical evolution of matter, radiation and dark energy for the model with double exponential potential, in which tracker mechanism was used to provide a non-negligible energy density at early epoch[17]. The dynamical evolution of equation of state w vs N are given in Figure 5.

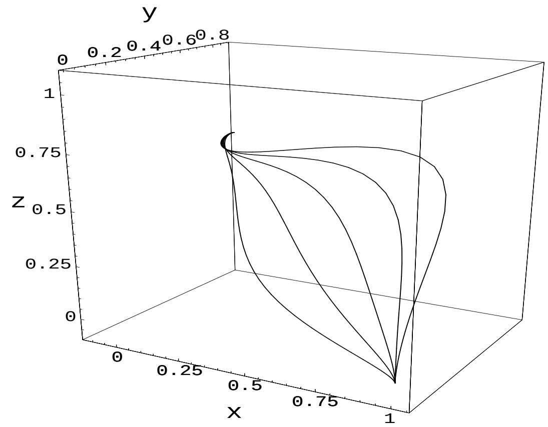

FIG. 3: The attractor property of the quintessence at the presence of dust matter. We choose $\alpha_1 = -1$, $\alpha_2 = 1$, $\gamma = 1$, $\kappa = 1$. The heteroclinic orbit connects the critical point which corresponds to the case (v) to the kinetic-dominated critical point $(1, 0, 0)$.

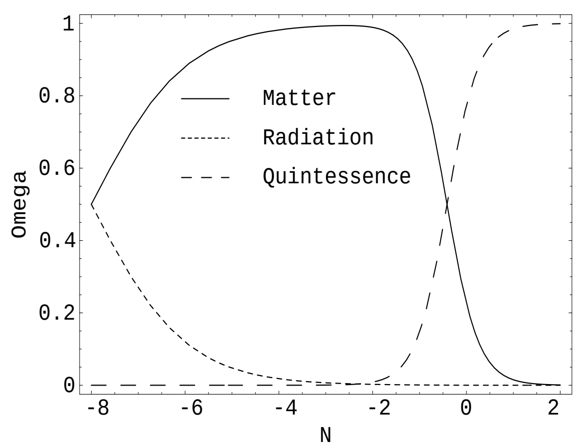

FIG. 4: Evolution of cosmic parameters for matter, radiation and quintessence. Plot begins from the equipartition epoch.

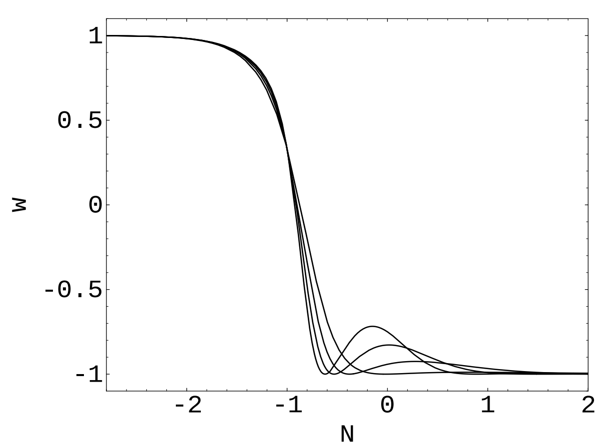

FIG. 5: The evolution of the equation of state w vs N. We choose $\alpha_1 = -2, -1.5, -1, -0.5$ respectively and $\alpha_2 = 1$, $\gamma = 1$, $\kappa = 1$.

IV. CONCLUSION AND DISCUSSION

In this paper, we investigate the cosmological dynamics of scalar field with a double exponential potential. We show that the late-time asymptotic behavior does not always corresponding to the single-exponential case. Furthermore, we study varied heteroclinic orbits including the tracking one. Tracking behavior consists in the possibility to avoid the fine tuning of initial conditions. The numerical calculation show that the tracking orbits have the similar but not exactly equal dynamical behavior and the initial possibilities consist in a wide range.

Why there are new features for double exponential potential? First of all, the parameter Γ here is dependent on the scalar field in contrast with single case. The second reason is that new features will occur in the regime where both exponential terms are remarkable. The third possible reason is that the heteroclinic orbits connect not only late-time but also early-time behavior.

We can extend our discussion to the $\mathrm{O}(N)$ quintessence model[5], the initial energy density could be larger and would not affect the evolution too much because the 'angular' kinetic energy decreases with a^{-b}. In a scalar field model, tracker mechanism was used to provide a non-negligible energy density at early epoch. In the $\mathrm{O}(N)$ quintessence model, if equipped with the same tracker mechanism, it will admit a wider range of initial energy density[17].

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